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Numerical Design of Transonic Cascades*

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The method developed by Bauer, Garabedian, and Korn ("Supercritical wing Sections," Springer-Verlag, Berlin, 1975) for the design of wing sections that exhibit shock-free flow at high subsonic speeds has been applied to the problem of cascade design. A cascade is a periodic distribution of two-dimensional blade cross sections that serves as a basis for the design of axial flow compressors and turbines. Those aspects of the design procedure that are used for both airfoil and cascade design are reviewed briefly in this paper. Emphasis has been placed on the differences between airfoil and cascade design. In particular, the form of the singular solution in the hodograph plane is different in the two cases.

1. INTRODUCTION

In recent years, there has been a growing concern about the availability and utilization of our energy resources. Considerable effort has gone into the development of new resources and into methods of more efficient use of existing energy supplies. The procedure described here can be used to design more efficient high speed turbomachinery which could result in significant fuel conservation. In particular, this technology could be the basis for the design of improved axial flow compressors used for the production of enriched uranium at gaseus diffusion plants. It can also be applied to the design of lighter weight and more efficient airplane engines.

In this paper, the technique developed by Bauer *et al.* [2] for isolated airfoil design is applied to the problem of designing transonic turbine and compressor blades found in axial flow compressors. The method is applicable to two-dimensional problems. Therefore, the model for design will be a periodic distribution of two-dimensional blade cross sections called a cascade. As the distance between blades increases, the solution must approach that of the isolated airfoil; thus this paper can be viewed as a generalization of the airfoil design method.

As with the airfoil, the problem reduces to finding smooth, transonic steady state solutions to the equations of irrotational motion of an inviscid compressible fluid about some object. In this case, the object will be a periodic distribution of blades. Since the equations of motion are the same as those for the airfoil design problem,

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the same technique of conjugate characteristic coordinates extended into the complex domain can be employed. A more detailed description of this method is found in [2].

The main differences between designing a compressor blade and an airfoil are the locations and types of singularities of the solution in the hodograph plane. In the airfoil case, the basic singularity is a pole in the hodograph plane at some prescribed free stream velocity. For the cascade problem the velocity far upstream of the cascade, called the inlet velocity, is different from the velocity far downstream of the cascade, called the exit velocity. In fact, the function of the cascade is to change the inlet velocity in some prescribed way. Thus, in the hodograph plane there are two velocities that correspond to infinity in the physical plane. The appropriate singularities in this case are logarithms at the inlet and exit velocities. As the distance between these logs decreases, the spacing between blades increases, approaching the isolated airfoil case. The logarithmic singularities behave like a dipole; in the limit they approach the correct singularity for flow around an airfoil.

For the cascade design problem as with the airfoil design problem, the inverse problem is solved; that is, solutions to the equations of motion with the appropriate singularity are generated and the streamline passing though the stagnation point is examined. This streamline describes the shape of the body. To generate solutions to the flow equations a set of parameters is chosen which defines physical quantities and describes an initial analytic function. The shape of the streamline in the hodograph plane corresponding to the blade is also prescribed. The program finds a solution to the equations of motion whose stagnation streamline matches the prescribed one in a least squares sense. If this streamline representing the body is not closed, or if it is not physically realizable in the physical plane (e.g., negative thickness), the input parameters must be changed and the program rerun.

In Section 2, the equations of motion are presented and rewritten in characteristic coordinates that are extended into the complex domain. A method to generate solutions with the appropriate singularity for a cascade is formulated in Section 3. Section 4 discusses the problem of choosing the initial function. Section 5 presents a description of the numerical technique that has been programmed on a CDC 6600 computer. The significant modifications of the design procedure of [1] are discussed in Section 6 and some results are given.

The procedure presented in this paper enables the design of supercritical blades. This offers many advantages over conventional blades. Cascades can be designed for which each blade is highly loaded, thereby reducing the number of blades necessary to achieve a desired amount of compression and turning. In addition to saving construction costs, lower loss coefficients can be expected since there are fewer blades. For airplane engines the weight savings mean an additional fuel savings. Alternately, compressor blades can be designed with higher inlet Mach numbers and more compression before the onset of drag rise than conventional blades. This should make it possible to reduce the number of stages, again reducing costs and saving fuel.

2. Equations of Motion

Let x and y be Cartesian coordinates and let u and v be the corresponding velocity components. Then the steady state equation of motion for an inviscid polytropic gas with isentropic, irrotational flow is

$$(c^2 - u^2)\phi_{xx} - 2uv\phi_{xy} + (c^2 - v^2)\phi_{yy} = 0, \tag{2.1}$$

where ϕ is the velocity potential, $\phi_x = u$, $\phi_y = v$, and c is the local speed of sound.

Let p denote pressure and ρ density. Then c is defined by

$$c^2 \equiv dp/d\rho = \gamma(p/\rho), \tag{2.2}$$

where γ is the gas constant. Conservation of energy is expressed by Bernoulli's equation

$$c^2 = c_0^2 - [(\gamma - 1)/2]q^2; \qquad q^2 = u^2 + v^2, \tag{2.3}$$

where c_0 is the sound speed at stagnation velocity. Equation (2.3) shows that c, hence ρ and p, are functions only of the speed q.

Introducing characteristic coordinates ξ and η, Eq (2.1) can be written as a system of four linear equations for u, v, x, and y (cf. [2]). Computing the characteristic directions

$$\frac{dv}{du} = \lambda_\pm = \frac{uv \pm c(q^2 - c^2)^{1/2}}{c^2 - v^2} \tag{2.4}$$

we can write the equations of motion in characteristic form as follows:

$$v_\xi = \lambda_+ u_\xi; \qquad v_\eta = \lambda_- u_\eta, \tag{2.5a}$$

$$\lambda_- y_\xi + x_\xi = 0; \qquad \lambda_+ y_\eta + x_\eta = 0. \tag{2.5b}$$

The local Mach number M is defined as q/c. The equations of motion are hyperbolic when $\lambda_\pm$ are real and unequal, which occurs when the flow is supersonic, $M > 1$. For $M < 1$, the flow is subsonic and the complex conjugate roots in $\lambda_\pm$ mean that the equations of motion are elliptic. The set of points where $M = 1$ is called the sonic line and is a circle in the hodograph plane $q =$ constant. This constant is called the critical speed, denoted by q_*.

Since we are interested in solutions to these equations in a region of the hodograph plane where both subsonic and supersonic flow may occur, a method applicable to both regions is necessary. The method of complex extension [4] is ideally suited since, in the complex domain, there is no distinction of type; in fact, $\lambda_\pm$ need not be real or complex conjugate. The only theoretical difficulty that arises in applying complex extension is along the set of points where $\lambda_+ = \lambda_-$, i.e., $M = 1$, since Eqs. (2.5b) are coincident there. This problem is easily treated (cf. [2. 10]).

With the method of complex extension, the independent variables ξ and η are each extended into the complex domain so the solution u, v, x, and y must be allowed to take on complex values. Since both ξ and η have real and imaginary components, this amounts to enlarging the independent domain from two dimensions to four dimensions.

The solution should be computed in as small a subset of the extended space as is necessary to obtain the solution in the real hodograph plane. It is possible to choose characteristic coordinates ξ and η so that the set of points $\xi = \bar{\eta}$ includes the real subsonic hodograph plane. These characteristic coordinates are called conjugate characteristic coordinates and are described in [2].

Equations (2.5a) are independent of x and y; therefore, $u(\xi, \eta)$ and $v(\xi, \eta)$ depend only on the selection of characteristic coordinates, not on the solution to Eqs. (2.5b). ξ and η may be viewed as generalized hodograph variables. In fact, closed form expressions for conjugate characteristic coordinates in terms of the hodograph variables u and v can be derived by integrating (2.4) (cf. [13]). It follows that s and s^* are conjugate characteristic coordinates by choosing

$$s = h(q)e^{-i\theta}; \qquad s^* = h(q)e^{i\theta}, \tag{2.6}$$

where

$$h(q) = Kq\{[c - (c^2 - q_u^2)^{1/2}]^{1/\mu}[(c^2 - q^2)^{1/2} + c]\}^{-1} \tag{2.7}$$

and $\mu^2 = (\gamma - 1)/(\gamma + 1)$, $\theta = \tan^{-1} v/u$ and K is a constant determined so that $h(1) = 1$. As the critical speed q^* becomes infinite $h(q)$ approaches q. Thus, for incompressible flow s approaches $w = u - iv$ and s may therefore be viewed as an analog of the complex velocity for compressible flow. The function h is real when q is real and subsonic, so that s and s^* are complex conjugates in the subsonic region. Let ξ and η be complex characteristic coordinates and consider mappings of the form

$$s = F(\xi); \qquad s^* = \overline{F(\bar{\eta})} \tag{2.8}$$

where F is an arbitrary analytic function. For any function F, ξ and η are characteristic coordinates. Also, it follows from (2.8) that whenever $s = \bar{s}^*$ then $\xi = \bar{\eta}$ so that ξ and η are also conjugate coordinates for any analytic function F.

In previous work on airfoil design it was sufficient to select a two parameter family of transformations

$$s = F(\xi) = 1 - 2B\xi - \xi^2 \tag{2.9}$$

for the complex parameter B that was suggested by considering incompressible flow around an elliptic cylinder. This led to a class of solutions that had a single branch point in the hodograph plane at $\xi = -\phi$. Moreover, design in the ξ-plane is simpler than in the W-plane because a simple closed curve representing the body streamline is transformed into the two-sheeted hodograph. Turbine and compressor blades generally have a high degree of camber and may require more than one branch point. Therefore, it is convenient to let

$$s = 1 + \sum_{k=1}^{l} c_k \xi^k \tag{2.10}$$

which can have as many as $l - 1$ branch points. Locating the branch points can be done by examining their location for incompressible flow around a cambered Joukowski profile.

Work is presently underway to develop an effective method for the selection of the function F to achieve the desired design objectives [6]. The selection of a suitable transformation F for turbine design will be the subject of a future report [11].

3. Singular Solution for the Cascade Problem

The method of complex extension can be applied to Eqs. (2.5) to generate analytic solutions of the characteristic initial value problem in the extended hodograph plane. Let $\xi = \xi_C$ and $\eta = \eta_C$ be two characteristic initial planes and assign data for either x or y along each of them. The pair of ordinary differential equations along each characteristic given by (2.5b) can be integrated to get the other dependent variable. The solution at points not on either characteristic initial plane can be found by the numerical technique described in Section 5. The solution will be analytic provided the initial data are analytic.

Flow through a cascade requires a solution that has an inlet velocity at $-\infty$ and an exit velocity at $+\infty$ in the physical plane. Let $w_B = u_B - iv_B$ and $w_A = u_A - iv_A$ denote the complex upstream and downstream velocities, respectively. Let (ξ_B, η_B) and (ξ_A, η_A) be the corresponding values of the characteristic coordinates found by solving (2.6) and (2.8) once the transformation function F has been prescribed. Since ξ and η are conjugate coordinates and u_A, u_B, v_A, and v_B are real quantities, it follows that $\eta_A = \bar{\xi}_A$ and $\eta_B = \bar{\xi}_B$. The solution in the physical plane is required to have infinities at the two points $(\xi_A, \bar{\xi}_A)$ and $(\xi_B, \bar{\xi}_B)$, which implies that there is a source at ξ_B and a sink at ξ_A in the real hodograph plane. The solution must also be periodic in the physical plane. Any closed loop in the hodograph plane surrounding both ξ_A and ξ_B corresponds to a closed loop in the physical plane enclosing a blade. This enclosure requirement will be modified for the wake model of the trailing edge.

One can verify that a solution of the form

$$x = \mathrm{Re}[X^1 \log(\xi - \xi_A) + X^2 \log(\xi - \xi_B) + X^3], \tag{3.1a}$$

$$y = \mathrm{Re}[Y^1 \log(\xi - \xi_A) + Y^2 \log(\xi - \xi_B) + Y^3] \tag{3.1b}$$

satisfies the above requirements for suitable choices of the regular functions X^1, X^2, X^3, Y^1, Y^2, and Y^3. The assumption that X^3 and Y^3 are regular leads to the requirement that X^1, Y^1 and X^2, Y^2 be solutions to the partial differential equations (2.5b) with initial conditions

$$\lambda_- Y^1 + X^1 = 0, \tag{3.2a}$$

$$\lambda_- Y^2 + X^2 = 0 \tag{3.2b}$$

on $\xi = \xi_A$ and $\xi = \xi_B$, respectively. This can be verified by substituting Eqs. (3.1a, b) into (2.5b). From this construction it can be seen that X^1, Y^1 and X^2, Y^2 are essentially Riemann functions for Eqs. (2.5b).

Equations (2.5b) can be solved with initial conditions (3.2a, 3.2b) to obtain X^1 along $\xi = \xi_A$ and X^2 along $\xi = \xi_B$. X^1 and X^2, along the corresponding characteristics of the other family $\eta = \eta_A$, $\eta = \eta_B$, must be defined to obtain a well-posed characteristic initial problem for the determination of X^1, X^2, Y^1, and Y^2 everywhere. Computation will be saved by defining the data on the other family of characteristics by reflection, i.e., $X^1(\xi, \eta_A) = \overline{X^1(\xi_A, \bar{\xi})}$; $X^2(\xi, \eta_B) = \overline{X^2(\xi_B, \bar{\xi})}$. Reflection is not generally possible since X^1, X^2, Y^1, Y^2 need not be real at ξ_A, η_A and ξ_B, η_B. However, since Eqs. (2.5b) admit constant solutions, solutions can be found for X^1, X^2, Y^1, and Y^2 of the form

$$X^1 = \hat{X}^1 + iX_A\,; \qquad X^2 = \hat{X}^2 + iX_B\,, \tag{3.3a}$$

$$Y^1 = \hat{Y}^1 + iY_A\,; \qquad Y^2 = \hat{Y}^2 + iY_B\,, \tag{3.3b}$$

where X_A, X_B, Y_A, and Y_B are real constants and $\hat{X}^1$, $\hat{X}^2$, $\hat{Y}^1$, and $\hat{Y}^2$ are symmetric solutions (i.e., $\hat{X}^1(\xi, \eta) = \overline{\hat{X}^1(\bar{\eta}, \bar{\xi})}$, etc.) to Eqs. (2.5b) satisfying (3.2a),–(3.2b) along the initial characteristics.

The equations for X^3 and Y^3 obtained by inserting (3.1a), (3.2b) into (2.5b) are

$$\lambda_- Y_\xi^3 + X_\xi^3 = \frac{(\lambda_- \hat{Y}^1 + \hat{X}^1) + i(\lambda_- Y_A + X_A)}{\xi_A - \xi} + \frac{(\lambda_- \hat{Y}^2 + \hat{X}^2) + i(\lambda_- Y_B + X_B)}{\xi_B - \xi}, \tag{3.4a}$$

$$\lambda_+ Y_\eta^3 + X_\eta^3 = 0, \tag{3.4b}$$

where the regularity of the right-hand side of (3.4a) is assured by the initial conditions (3.2a), (3.2b). In order to formulate a well-posed problem for the evaluation of X^3 and Y^3 it is necessary to assign initial data along a pair of initial characteristics. Let $\xi = \xi_C$ and $\eta = \eta_C$ be initial characteristics whose intersection is a point in the real subsonic domain. This implies that $\eta_C = \bar{\xi}_C$. We assign

$$X^3(\xi_C, \eta) = G(\eta); \qquad X^3(\xi, \eta_C) = G(\eta_C) \tag{3.5}$$

on the initial characteristics $\xi = \xi_C$ and $\eta = \eta_C$, respectively, where G is an analytic function. The selection of G is the subject of Section 4. Since the solution for x and y

uses only the real parts of X^3 and Y^3 many combinations of initial data can be precribed that will yield the same real parts. In particular, the symmetric set

$$X^3(\xi_C, \eta) = \tfrac{1}{2}[G(\eta) + \overline{G(\eta_C)}]; \qquad X^3(\xi, \eta_C) = \tfrac{1}{2}[G(\eta_C) + \overline{G(\xi)}] \tag{3.6}$$

has been used to reduce computation in the subsonic portion of the flow.

The four real constants X_A, Y_A, X_B, and Y_B must be determined. First, consider the requirement that the solution be single valued for any real closed path encircling a blade;

$$\oint dx = \mathrm{Re}\left[\oint \left(\frac{\hat{X}^1 + iX_A}{\xi - \xi_A} + \frac{\hat{X}^2 + iX_B}{\xi - \xi_B}\right) d\xi\right] = -2\pi(X_A + X_B), \tag{3.7a}$$

$$\oint dy = \mathrm{Re}\left[\oint \left(\frac{\hat{Y}^1 + iY_A}{\xi - \xi_A} + \frac{\hat{Y}^2 + iY_B}{\xi - \xi_B}\right) d\xi\right] = -2\pi(Y_A + Y_B), \tag{3.7b}$$

where the path of integration is around both ξ_A and ξ_B in the plane $\xi = \bar{\eta}$. Therefore, to have a single-valued solution, $X_B = -X_A$ and $Y_B = -Y_A$. A further requirement is that the stream function ψ defined by

$$d\psi = (\rho u)\, dy - (\rho v)\, dx \tag{3.8}$$

be single valued for any similar path. Since ρu and ρv are real in the real domain it follows that

$$\oint d\psi = 2\pi[\rho_A u_A Y_A + \rho_B u_B Y_B - \rho_A v_A X_A - \rho_B v_B X_B] = 0. \tag{3.9}$$

Making use of the closure conditions (3.7a),-(3.7b) and (3.9) we have

$$(\rho_A u_A - \rho_B u_B) Y_A = (\rho_A v_A - \rho_B v_B) X_A\,. \tag{3.10}$$

From (3.1a), (3.1b) it is noted that the solution is periodic with a period of X_B and Y_B in x and y, respectively. The stagger angle β and the gap g are defined by

$$\beta = \tan^{-1}(Y_A/X_A); \qquad g^2 = X_A{}^2 + Y_A{}^2. \tag{3.11}$$

Given the inlet and exit velocities, the stagger angle is computed from (3.10). Vertical stacking can be achieved by a plane rotation that transforms β into $\pi/2$. To have physical significance the gap must be measured relative to some suitable distance measurement such as the blade chord. Since this is not known until the solution has been found, the solution can be normalized by requiring that the gap g have magnitude $|\xi_A - \xi_B|^{-1}$. As a result of this normalization, the chord length will change slowly as $\xi_A \to \xi_B$. Moreover, in the limiting case, the isolated blade will have the same singularity as the one used in our previous work on airfoil design.

The wake model of the trailing edge that was formulated in [1] to correct for boundary layer displacement can easily be incorporated into this cascade design procedure. This is done by requiring that the residues of (3.7a), (3.7b) be nonzero. In fact, they should be proportional to X_B and Y_B, respectively, so that the blades will have a vertical gap when stacked vertically. The constant of proportionality is the ratio of the thickness of the trailing edge outside the boundary layer to the gap and is chosen to be small. Eqaution (3.10) has to be modified accordingly.

The problem is flow through a cascade of blades can, therefore, be solved by generating the solution of two characteristic initial value problems for the homogeneous equations (2.5b), one along $\xi = \xi_A$, $\eta = \bar{\xi}_A$, the other along $\xi = \xi_B$, $\eta = \bar{\xi}_B$, and by solving a characteristic initial value problem for the inhomogeneous equations (3.4a), (3.4b) for arbitrary data chosen along some initial characteristics $\xi = \xi_C$, $\eta = \eta_C$. The initial data for the homogeneous equations are determined by solving ordinary differential equations along the initial characteristics reminiscent of the Riemann functions. The four constants X_A, X_B, Y_A, and Y_B are completely determined by three closure conditions and normalization. The solution for x and y is formed from the solution of these three problems using (3.1a), (3.1b) and has the singularities necessary for flow through a cascade.

4. The Selection of Initial Data

Solutions to the equations of motion of the form (3.1a),–(3.1b) will have the appropriate singular solution for flow through a cascade. However, arbitrary initial data assigned along the characteristic initial planes, $\xi = \xi_C$ and $\eta = \eta_C$, will not necessarily generate solutions with a physical interpretation. Once the transformation from the hodograph variables to characteristic coordinates is known, the design problem is mainly that of selecting the initial data to achieve as many design goals as possible. Many design objectives, such as boundary layer control, cannot be stated in precise terms within the context of this inviscid model. Other objectives may be impossible to achieve within the class of shockless solutions. Thus, the problem of choosing initial data requires experimentation.

The shape of the blade is found by examining the level curve $\psi = \psi_0$ of a solution where ψ_0 is the value of the stream function at stagnation. The shape of this curve will depend upon the initial data and, in fact, does not have to be a closed curve. A procedure for selecting the initial data that attempts to minimize the chances of obtaining extraneous solutions is outlined. Most of the ideas used for the airfoil design problem described in [1, 2] are applicable. Let the initial function be of the form

$$G(\eta) = \sum_{j=1}^{N} A_j g_j(\eta), \tag{4.1}$$

where A_j are constants and g_j are simple functions.

The linear parameters A_j of Eq. (4.1) are found by a least squares process. Instead of prescribing the linear parameters, a curve or set of arcs in the hodograph plane is selected on which the blade is to lie. The linear constants in (4.1) are determined so that the solution it generates minimizes

$$\int W\psi^2 \, d\xi, \tag{4.2}$$

where the integration is taken along the prescribed arcs. The weighting function W can be adjusted to achieve a better fit along any given arc.

Prescribing the automation arcs in the characteristic planes is in effect prescribing the body streamline in the hodograph plane which is convenient for boundary layer control. Boundary layer separation can be eliminated, or delayed until the last percentage of chord in most cases, by prescribing an arc that approaches the trailing edge along a line of constant speed.

Additional constraints are placed on the solution. The body streamline is required to pass through a designated point in the hodograph plane corresponding to the trailing edge, and the derivatives ψ_u, ψ_v are required to vanish there. This is our model of the Kutta–Joukowski condition in the hodograph plane.

The above constraints and the least-squares conditions greatly reduce the effort in selecting meaningful initial data. The problem as stated is overdetermined and, therefore, good agreement cannot be expected between the prescribed arcs and the resulting streamline unless they are selected in a manner consistent with the given compression and turning.

5. Numerical Method of Solution

The finite difference technique described in [2] for airfoil design has been applied with minor modifications for the cascade design problem. In the real domain this technique is called the method of characteristics or Massau's method [3] and is essentially a predictor–corrector scheme. The method is applied in the complex domain merely by performing all operations using complex arithmetic.

The solutions for X^1, Y^1, X^2, Y^2, and X^3, Y^3 must be found. The functions X^1, Y^1 and X^2, Y^2 satisfy the equations of motion (2.5b) with data given by (3.2a), (3.2b) on the initial characteristics, while X^3, Y^3 satisfy the inhomogeneous equations (3.4a), (3.4b) whose right-hand side is computed from X^1, Y^1 and X^2, Y^2. The method of Massau will be applied to each of these three sets of unknowns. The hodograph variables may be obtained explicitly from Eqs. (2.8), (2.6), and (2.7). However, Massau's method with characteristic initial data obtained from Eqs. (2.6)–(2.8) is faster.

The initial value problems for X^1, Y^1 and X^2, Y^2 start on initial characteristics ξ_A, η_A and ξ_B, η_B, respectively. The initial value problem for X^3, Y^3 has initial characteristics ξ_C, η_C along which it is necessary to know X^1, Y^1, X^2, and Y^2 in order to evaluate the inhomogeneous terms. The application of this method requires the selection of a path along each of the initial characteristics, as illustrated in Fig. 1A. Each path starts at the point of intersection of the two initial characteristics. Let r and t be real parameters. We assign data along a path $\eta = \eta(t)$ from η_A on the initial characteristic $\xi = \xi_A$ and along another path $\xi = \xi(r)$ from ξ_A on the other initial characteristic. Given the initial data along these paths, application of the finite difference scheme determines the solution of the two-dimensional surface consisting of the points of intersection $\xi(r)$, $\eta(t)$. To find the solution at point ξ_P, η_P, two initial paths are chosen; one on $\xi = \xi_A$ from $\eta = \eta_A$ to $\eta = \eta_P$ and the other on $\eta = \eta_A$ from $\xi = \xi_A$ to $\xi = \xi_P$. For analytic initial data the solution at the point ξ_P, η_P is independent of path. The values of X^1, Y^1, X^2, Y^2 for the evaluation of the right-hand side of (3.4a) can be computed as they are needed by selecting each initial path on η_A so that it starts from $\xi = \xi_A$, passes through $\xi = \xi_B$, and then through $\xi = \xi_C$. Similarly, the initial path along $\xi = \xi_A$ begins at $\eta = \eta_A$ and passes through η_B and η_C. A typical pair of paths is shown in Fig. 1A. The corresponding finite difference grid is illustrated in Fig. 1B.

The solution throughout the four-dimensional domain can be computed by selecting appropriate pairs of paths. The solution in the real hodograph plane should be computed with as few pairs as possible. Complex conjugate sets of paths yield one line of the solution in the real subsonic domain. One set of paths determines a portion of the solution in the supersonic region bounded by two characteristics and the sonic line. Generally, this region contains the full supersonic portion of the flow outside the blade.

Each set of initial paths is described by the vertices of a polygonal arc and a set of grid points superimposed in such a way that each vertex is a grid point and that the maximum distance between adjacent grid points is less than some fixed amount. Let u_{jk} denote the value of $u(\xi(r_j), \eta(t_k))$, where r and t are real parametric representations of the initial paths. Note that $r = 0$ and $t = 0$ are the initial curves on $\xi = \xi_A$ and $\eta = \eta_A$, respectively. The predictor step of the method of characteristics for the solution of Eq. (2.5a) at the point j, k yields the two simultaneous linear uquations

$$\tilde{v}_{j,k} - v_{j-1,k} = (\lambda_+)_{j-1,k}\,(\tilde{u}_{j,k} - u_{j-1,k}), \tag{5.1a}$$

$$\tilde{v}_{j,k} - v_{j,k-1} = (\lambda_-)_{j,k-1}\,(\tilde{u}_{j,k} - u_{j,k-1}) \tag{5.1b}$$

for the predicted values $\tilde{u}_{j,k}$ and $\tilde{v}_{j,k}$ which are first-order-accurate estimates of u and v. The initial values of u and v along $j = 0$ and $k = 0$ are determined explicitly from the hodograph transformation (2.6)–(2.8).

The values resulting from the predictor step are used to compute first-order-accurate midpoint values for $\lambda_\pm$ given by

$$\tilde{\lambda}_+ = \tfrac{1}{2}[(\lambda_+)_{j-1,k} + \lambda_+(\tilde{u}_{j,k}, \tilde{v}_{j,k})], \tag{5.2a}$$

$$\tilde{\lambda}_- = \tfrac{1}{2}[(\lambda_-)_{j,k-1} + \lambda_-(\tilde{u}_{j,k}, \tilde{v}_{j,k})]. \tag{5.2b}$$

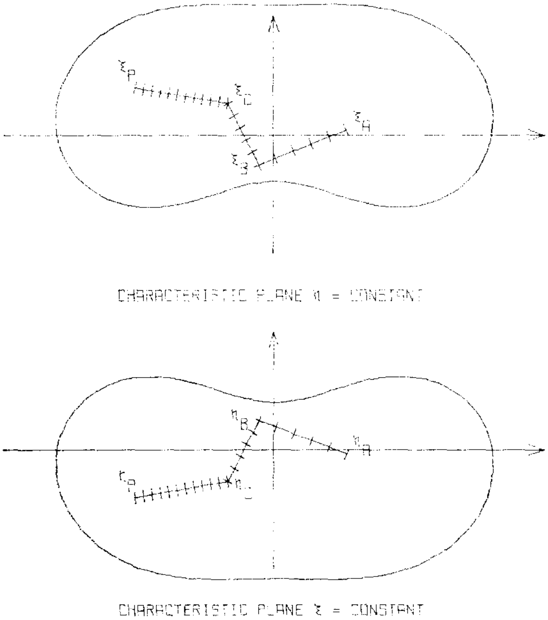

FIGURE 1A

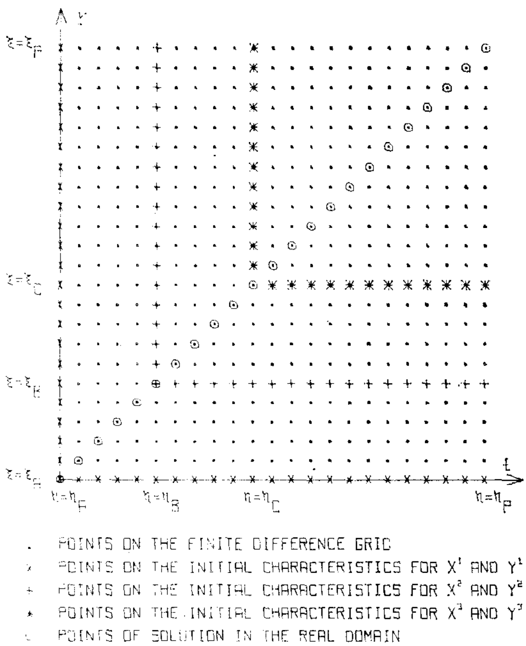

FIGURE 1B

For the corrector step

$$v_{j,k} - v_{j-1,k} = \tilde{\lambda}_+(u_{j,k} - u_{j-1,k}), \tag{5.3a}$$

$$v_{j,k} - v_{j,k-1} = \tilde{\lambda}_-(u_{j,k} - u_{j,k-1}) \tag{5.3b}$$

are solved for $u_{j,k}$ and $v_{j,k}$ to obtain second-order accuracy.

A similar set of equations is solved to obtain X^1, Y^1, X^2, Y^2, and X^3, Y^3 to second order accuracy. Equations (3.1a)–(3.1b) are evaluated at points in the real hodograph to obtain second-order-accurate values of x and y. The stream function is obtained by applying a second-order-accurate difference approximation to (3.8). Points on the blade are computed from the stream function by inverse parabolic interpolation.

The least squares automation outlined in Section 4 is accomplished by generating solutions to Eqs. (3.4a), (3.4b) along the prescribed automation arcs with varying initial data. For each linear parameter A_j of function G, given by (4.1), initial data are assigned which have that particular parameter set to one and all the others set to zero. This set of solutions determines the coefficients of a linear system of equations for A_j as described in [2]. The desired initial function G is given by the solution to this system of equations.

The systems of two simultaneous linear equations (5.1a), (5.1b) and (5.3a), (5.3b) become ill-conditioned whenever λ_+ and λ_- are nearly equal, i.e., near points of the complex sonic line; therefore, our largest errors occur in the vicinity of the sonic line.

A third-order accurate method can be obtained by Richardson's extrapolation. This option has been incorporated into the computer program to obtain more accurate resolution in the vicinity of the leading and trailing edges. We cannot expect improvement near the sonic line because of the singularity of the hodograph transformation there.

6. Refinements to the Design Procedure and Results

A number of improvements have been made in the cascade design program that have not yet been incorporated in the airfoil design procedure (cf. [2]). The most significant modification was made to reduce execution time. Blades with low gap/chord require more computation time since the length of the path connecting the inlet and exit velocities in the characteristic plane increases inversely with the gap to chord ratio. The least squares calculation takes most of the computing time since a solution for X^3 and Y^3 along each automated path must be computed for each parameter A_j.

A significant reduction in execution time can be achieved by computing X^3 and Y^3 for each parameter A_j at the same time, because the righ-hand side of (3.4a) and $\lambda_\pm$ would only be computed once. For fine grids with many linear parameters, the required storage would make this procedure prohibitive. A compromise which allows for a fixed number of solutions of X^3 and Y^3 to be computed at each pass has been incorporated. A typical run with 10 of the A_j parameters obtained at each pass has decreased execution time for the automation by a factor of about 5 with less than twice the storage required. Additional coding improvements which make the program easier to operate will not be discussed.

A computer program for cascade design has been written that incorporates these ideas, and a few designs have been generated. A successful design required several runs, but the time for each run is small so that computing time is not prohibitive. It takes about $\frac{1}{2}$ min of central processor time on the CDC 6600 for a coarse mesh run. An intermediate grid and a fine grid take about 2 and 8 min, respectively. The third-order-accurate option increases computing time by a factor of about 5.

Figure 2 is a Calcomp plot of a blade which has been designed with this program in about 50 runs. All but the last few runs were made on a coarse grid. The inlet flow angle, when stacked vertically, is 45° and the turning angle is 29°. The gap chord ratio is 1.07. The Mach number distribution over the blade and the characteristics or

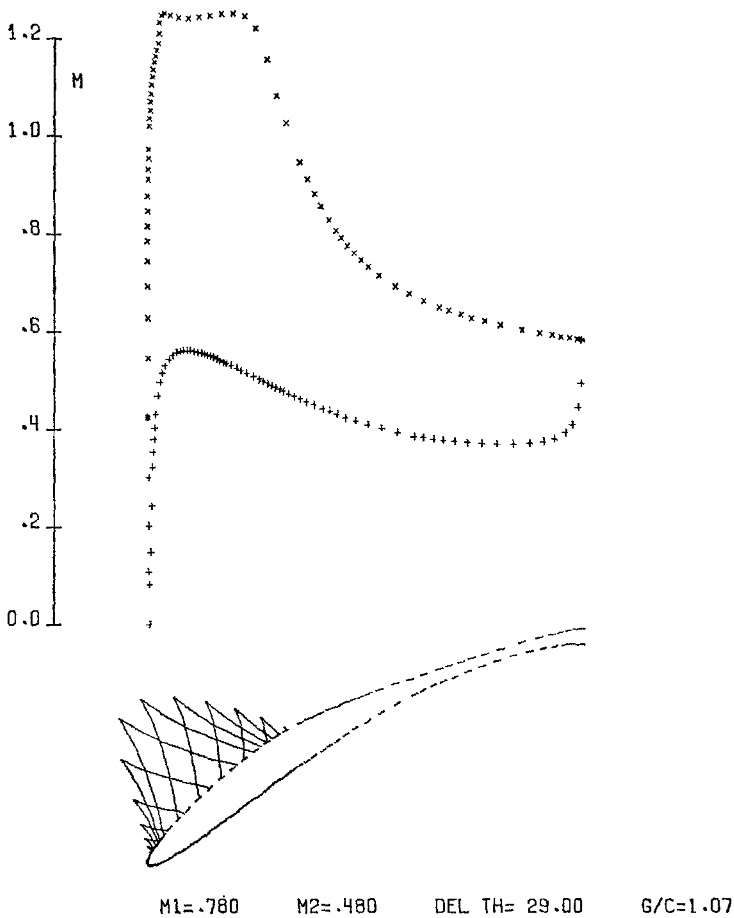

FIGURE 2

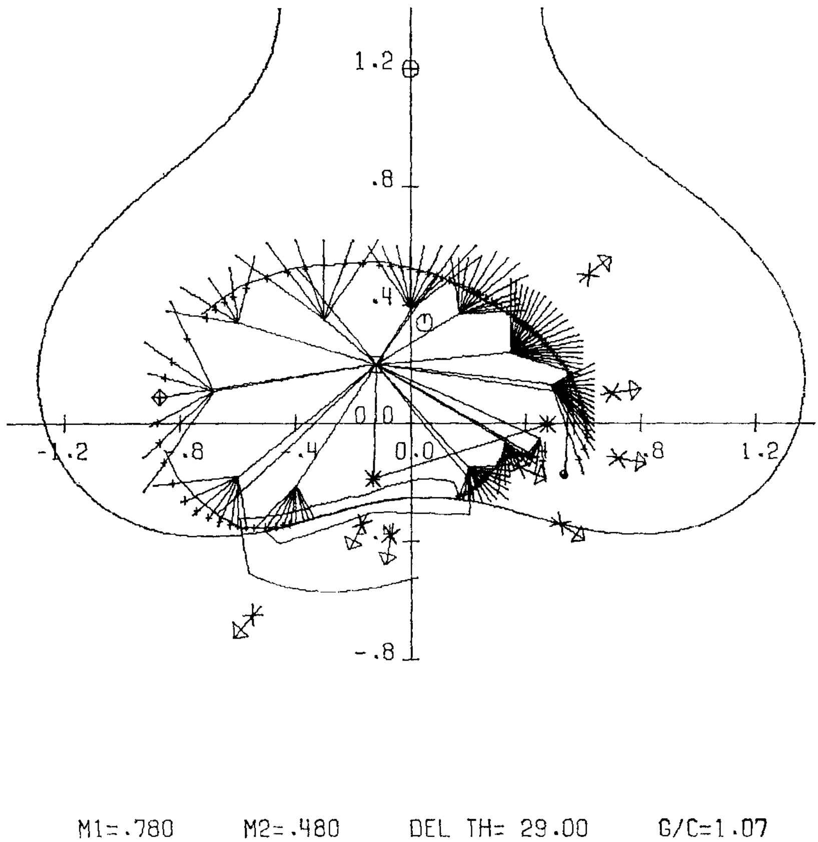

FIGURE 3

Mach lines in the supersonic region have been plotted. With this Mach number distribution no separation is predicted for Reynolds number 1.6 million.

The corresponding initial plane is illustrated in Fig. 3. The input paths and sonic locus have been plotted. The plus signs represent the preimages of points on the body, and the arrows illustrate the location of logarithmic singularities in the initial function and the direction of their cuts. An asterisk is placed at the points corresponding to the inlet and exit velocities. This figure is similar to those obtained previously with the airfoil design program (cf. [1, 2]).

A number of experiments performed on airfoils designed by this procedure (cf. [5, 7, 8, 9, 12]) have proven the usefulness of this method. In particular, Ref. [12] shows agreement with this theory even for Reynolds numbers less than 10^6. A blade designed by this technique has been tested in a cascade wind tunnel at the Pratt and Whitney Division of United Technologies.

Work to simplify the design procedure is in progress. A more detailed report on cascade design with emphasis on turbine design should appear shortly [11].

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